Fuzzy Subsethood for Type-n Fuzzy Sets

John T. Rickard
Distributed Infinity, Inc.
4637 Shoshone Drive
Larkspur, CO 80118
terry.rickard@gmail.com

Janet Aisbett and Greg Gibbon
Faculty of Science and IT
University of Newcastle
Callaghan, NSW, Australia
[janet.aisbett, greg.gibbon]@newcastle.edu.au

David Morgenthaler
Lockheed Martin Corp.
P.O. Box 277004 DC3535
Littleton, CO 80127
david.g.morgenthaler@lmco.com

***Abstract* –Subsethood is the degree of containment of one fuzzy set in another, usually expressed as a value in the unit interval. We extend Kosko's definition of subsethood to Type-n fuzzy sets, for which subsethood is a Type-(n-1) fuzzy set on the unit interval. We then show how to compute subsethood for general Type-2 and interval Type-3 fuzzy sets.**

I. Introduction

Zadeh [1] defined a subset of a fuzzy set as one whose membership function is dominated by the containing set. Kosko [2] allowed for degrees of subsethood of one fuzzy set in another, thus making subsethood itself a fuzzy notion. Several other definitions of subsethood have been proposed, along with suggested properties that candidate subsethood functions should possess [3, 4].

Subsethood has found many practical applications, e.g. [5, 6, 7]. In our work on conceptual space theory, subsethood has proved to be a fundamental operation used in defining fuzzy similarity of properties to properties, of observations to concepts, of concepts to concepts, and more [8, 9].

Because membership functions are themselves usually imprecise in practice, fuzzy sets were extended to Type-2 and higher order fuzzy sets by Zadeh [10]. For n > 1, Type-n fuzzy sets are sets whose membership functions take values in Type-(n-1) fuzzy sets on the unit interval. Such sets can model independent sources of uncertainty. For example, consider a universe of Pantone color samples. Uncertainty associated with the degree of whiteness of a particular sample may be due to uncertainty in what constitutes whiteness, but also to variations in the fabric on which the sample is printed, and to variations in lighting conditions. So "white" might be modelled as a Type-4 fuzzy set on the universe of color samples.

There has recently been renewed interest in application of Type-n fuzzy set theory, although most work has focussed on Type-2 fuzzy sets [11-15]. Given the importance of subsethood as a relationship between fuzzy sets, it is worthwhile to investigate how the notion might be defined for general Type-n fuzzy sets. Hence, Zadeh's extension principle is used here to generalize Kosko's definition of subsethood (section 2). Section 3 shows how to compute subsethood for Type-2 fuzzy sets, starting first with the interval Type-2 case. Section 4 deals with subsethood of interval Type-3 fuzzy sets, and section 5 gives examples.

Throughout the paper, $M(X)$ denotes the set of functions from X to the unit interval and $\int_X g(x)dx$ denotes the Lebesgue integral. Measurability is assumed of all sets and functions, a condition satisfied by typical real-world modelling on vector spaces and finite categorical domains.

II. Subsethood of Type-n fuzzy sets

A. Type-n fuzzy sets

For $n > 1$, the membership function of a Type-n fuzzy set, $\tilde{G}^{(n)}$, is a measurable mapping $\mu_{\tilde{G}^{(n)}} : X \to M\left([0,1]^{n-1}\right)$, called the primary membership function. $\mu_{\tilde{G}^{(n)}}(x)$ is called the secondary membership function at $x \in X$, and is a Type-(n-1) fuzzy set on the unit interval. The set of Type-n fuzzy set membership functions on X can also be identified with $M\left(X \times [0,1]^{n-1}\right)$.

Fig. 1 depicts a Type-3 fuzzy set, in which the secondary membership function is an interval Type-2 fuzzy set. That is, when considered as an element in $M\left(X \times [0,1]^2\right)$ the membership function takes values equal to unity over a footprint of uncertainty (FOU) bounded by continuous upper and lower bounding functions.

B. Fuzzy subsethood

Kosko [2] defined the *subsethood* $S(G,H)$ of a finite Type-1 fuzzy set G in a finite Type-1 fuzzy set H over the same universe X. That definition generalises to

$$S(G,H) = \frac{\int_X \min\left(\mu_G(x), \mu_H(x)\right) dx}{\int_X \mu_G(x)\,dx} \qquad (1)$$

where the integrals can be replaced by sums in the finite case. (When both integrals in (1) are zero, subsethood is calculated as the infimum of the ratio of the (Riemannian) upper sums.)

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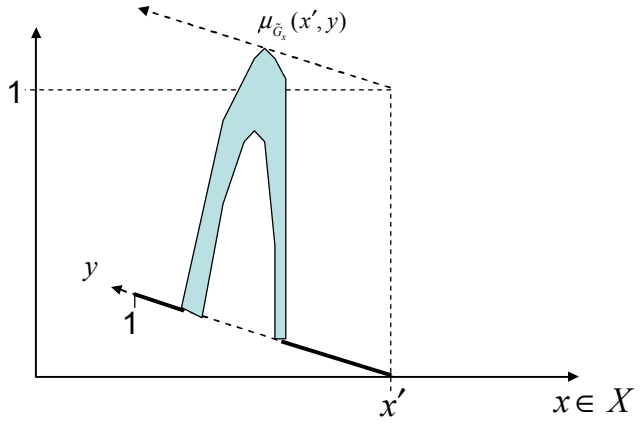

Fig. 1. Illustration of an interval Type-3 fuzzy set having a Type-2 membership function $\mu_{\tilde{G}_x}(y) = \mu_{\tilde{G}}(x,y)$ defined at $x' \in X$.

Subsethood is a function on pairs of membership functions rather than on the underlying universe *X*. That is,

$$S(G,H) \equiv f\left(\mu_G, \mu_H\right) \tag{2}$$

for $f : M(X) \times M(X) \to [0,1]$ defined as

$$f(u,v) = \int_X \min\left(u(x), v(x)\right) dx \Big/ \int_X u(x)\, dx . \tag{3}$$

A Type-n fuzzy set $\tilde{G}^{(n)}$ on *X* induces a Type-(n-1) fuzzy set on $M(X)$, through the membership function

$$\mu(u) = \inf_{x \in X} \left\{ \mu_{\tilde{G}_x^{(n)}}\left(u(x)\right) \right\}.$$

We extend definition (1) to Type-2 fuzzy sets by applying Zadeh's Extension Principle [10] to the bivariate function *f* on $M(X)$ defined in (3). This application gives

$$\mu_{S\left(\tilde{G},\tilde{H}\right)}(z) = \sup_{f(u,v)=z} \inf_{x \in X} \left\{ \mu_{\tilde{G}_x}\left(u(x)\right), \mu_{\tilde{H}_x}\left(v(x)\right) \right\} \tag{4}$$

for $u, v \in M(X)$ in the support of *f*.

On Type-3 fuzzy sets, $\inf_{x \in X} \left\{ \mu_{\tilde{G}_x^3}\left(u(x)\right), \mu_{\tilde{H}_x^3}\left(v(x)\right) \right\}$ is the intersection of fuzzy sets (possibly an infinite number of fuzzy sets). By the Extension Principle this intersection is the fuzzy set on the unit interval with membership function

$$\sup_{\inf\left(u'(x),v'(x)\right)=z'} \inf_{x \in X} \left\{ \mu_{G_x}\left(u(x),u'(x)\right), \mu_{H_x}\left(v(x),v'(x)\right) \right\}.$$

Substituting into (4) yields

$$\mu_{S\left(\tilde{G}^{(3)},\tilde{H}^{(3)}\right)}(z,z') = \sup\{ \inf_{x \in X} \left\{ \mu_{\tilde{G}_x^3}\left(u(x),u'(x)\right), \mu_{\tilde{H}_x^3}\left(v(x),v'(x)\right) \right\} : f(u,v) = z, \inf\nolimits_{x \in X}\left(u'(x),v'(x)\right) = z' \}.$$

Recursive application of the Extension Principle gives the expression for subsethood for general Type-n fuzzy sets. Using vector notation for conciseness, for any $\mathbf{z} \in [0,1]^{n-2}$ and $z \in [0,1]$, and for $u, v \in M(X)$ and $\mathbf{u}, \mathbf{v} \in M(X)^{n-2}$,

$$\mu_{S\left(\tilde{G}^{(n)},\tilde{H}^{(n)}\right)}(z,\mathbf{z}) = \sup_{\substack{f(u,v)=z \\ \inf_{x \in X}(\mathbf{u}(x),\mathbf{v}(x))=\mathbf{z}}} \inf_{x \in X} \left\{ \mu_{\tilde{G}_x^{(n)}}\left(u(x),\mathbf{u}(x)\right), \mu_{\tilde{H}_x^{(n)}}\left(v(x),\mathbf{v}(x)\right) \right\}. \tag{5}$$

The following proposition follows easily:

Proposition If $\tilde{G}^{(n)}$ and $\tilde{H}^{(n)}$ are interval Type-n fuzzy sets, then the subsethood of $\tilde{G}^{(n)}$ in $\tilde{H}^{(n)}$, $S\left(\tilde{G}^{(n)},\tilde{H}^{(n)}\right)$, is an interval Type-(n-1) fuzzy set.

III. Subsethood of Type-2 Fuzzy Sets

A. Interval Type 2 sets

Suppose that Type-2 fuzzy sets $\tilde{G}$ and $\tilde{H}$ have secondary membership functions $\mu_{\tilde{G}_x}(z) = 1$ for all $z \in \left[u_0(x), u_1(x)\right]$ and $\mu_{\tilde{H}_x}(z) = 1$ for all $z \in \left[v_0(x), v_1(x)\right]$, otherwise zero. From (4), subsethood has membership function $\mu_{S\left(\tilde{G},\tilde{H}\right)}(z) = 1$ for all z for which there exist $u, v \in M(X)$ satisfying the conditions:

$$f(u,v) \equiv \int_X \min\left(u(x), v(x)\right) dx \Big/ \int_X u(x)\, dx = z$$

and

$$\forall x \in X, u(x) \in [u_0(x), u_1(x)],\ v(x) \in [v_0(x), v_1(x)].$$

We know that $\mu_{S\left(\tilde{G},\tilde{H}\right)}(z) = 1$ over some interval $[s_\ell, s_r] \subseteq [0,1]$; we now compute the endpoints of this interval. Define

$$I_0 = \left\{ x \in X : v_1(x) \le u_0(x) \right\}$$

and

$$I_1 = \left\{ x \in X : v_0(x) \ge u_1(x) \right\}.$$

Then

$$f(u,v) = \frac{\int_{I_0} v(x)\, dx + \int_{I_1} u(x)\, dx + \int_{X - I_0 \cup I_1} \min\left(u(x), v(x)\right) dx}{\int_X u(x)\, dx}. \tag{6}$$

Since $v_0(x) \le v(x) \le v_1(x)$, we have

$$\begin{aligned} &\min\left(v(x), v_0(x)\right) \le \min\left(u(x), v(x)\right) \\ &\le \min\left(u(x), v_1(x)\right),\ x \in X \end{aligned}. \tag{7}$$

Substituting these inequalities into (6) yields the following upper and lower bounds of $f(u,v)$:

$$\frac{\int_{I_0} v_0(x)dx + \int_{I_1} u(x)dx + \int_{X-I_0\cup I_1} \min\left(u(x),v_0(x)\right)dx}{\int_X u(x)dx} \le f(u,v)$$

$$\le \frac{\int_{I_0} v_1(x)dx + \int_{I_1} u(x)dx + \int_{X-I_0\cup I_1} \min\left(u(x),v_1(x)\right)dx}{\int_X u(x)dx}. \quad (8)$$

We want to find the minimum value of the lower bound and the maximum value of the upper bound in order to determine the interval taken by $S(\tilde{G},\tilde{H})$. To find the lower bound, first define

$$I_<(u) = \left\{x\in X : u(x) < v_0(x), x\notin I_0\cup I_1\right\}$$
$$I_>(u) = \left\{x\in X : u(x) > v_0(x), x\notin I_0\cup I_1\right\},$$

so that

$$\min\left(u(x),v_0(x)\right) = u(x), x\in I_<(u),$$
$$\min\left(u(x),v_0(x)\right) = v_0(x), x\in I_>(u).$$

Now for each $x\in X$ we can find a measurable set B_x of X such that

$$B_x \subset I_0 \text{ if } x\in I_0; B_x \subset I_1 \text{ if } x\in I_1$$
$$B_x \subset I_<(u) \text{ if } x\in I_<(u); B_x \subset I_>(u) \text{ if } x\in I_>(u).$$

The lower bound in (8) can thus be considered to be the following function of $\lambda(x)\equiv\int_{B_x} u(x)dx$:

$$\frac{K(x)\lambda(x)+Y(x)}{\int_{X-B_x} u(x)dx+\lambda(x)}$$

where

$$Y(x) = \int_{I_0\cup I_>(u)} v_0(x)dx + \int_{I_1\cup I_<(u)-B_x\cap I_1\cup I_<(u)} u(x)dx \text{ and}$$
$$K(x) = \begin{cases} 0 & x\in I_0 \text{ or } x\in I_>(u) \\ 1 & x\in I_1 \text{ or } x\in I_<(u) \end{cases}. \quad (9)$$

The derivative with respect to $\lambda(x)$ of the lower bound is therefore

$$\frac{K(x)\left(\int_{X-B_x} u(x)dx+\lambda(x)\right)-Y(x)}{\left(\int_{X-B_x} u(x)dx+\lambda(x)\right)^2}$$

$$=\frac{K(x)\int_X u(x)dx - Y(x)}{\left(\int_X u(x)dx\right)^2}.$$

From (9) it follows that the lower bound is

$$\frac{-Y(x)}{\left(\int_X u(x)dx\right)^2}$$
$$=\frac{-\left[\int_{I_0} v_0(x)\,dx+\int_{I_1} u(x)\,dx+\int_{I_1}\min\left(u(x),v_0(x)\right)dx\right]}{\left(\int_X u(x)\,dx\right)^2} \quad (10)$$

when $x\in I_0\cup I_>(u)$, and

$$\frac{\int_X u(x)dx - Y(x)}{\left(\int_X u(x)dx\right)^2}$$
$$=\frac{\left[\int_{I_0}\left(u(x)-v_0(x)\right)dx+\int_{X-I_0\cup I_1}\left(u(x)-\min\left(u(x),v_0(x)\right)\right)dx\right]}{\left(\int_X u(x)dx\right)^2} \quad (11)$$

when $x\in I_1\cup I_<(u)$.

The lower bound on $f(u,v)$ therefore exhibits intervals of monotonic behaviour as a function of $\lambda(x)$. In the cases $x\in I_0$ or $x\in I_>(u)$, the lower bound *decreases* monotonically with increasing $\lambda(x)$, and thus achieves its minimum value when $u(x)=u_1(x)$ (more precisely, when $u(x')=u_1(x')$ for all x' in B_x). In the case $x\in I_1$, the lower bound *increases* monotonically with increasing $\lambda(x)$ and thus achieves its minimum value when $u(x)=u_0(x)$. In the case $x\in I_<(u)$ the lower bound increases monotonically with increasing $\lambda(x)$ up to a value $u(x)=v_0(x)$, but then decreases monotonically with further increases in $\lambda(x)$. Therefore its minimum will occur for one of the two values $u(x)=u_0(x)$ or $u(x)=u_1(x)$, but unfortunately where the minimum occurs cannot always be determined analytically.

Repeating these computations for the derivative of the upper bound in (8) yields expressions analogous to those in

(11) with $v_0(x)$ replaced by $v_1(x)$. In the cases $x \in I_0$ or $x \notin I_0 \cup I_1; v_1(x) > u(x)$, the upper bound *decreases* monotonically with increasing $\lambda(x)$ and thus achieves its maximum value when $u(x) = u_0(x)$ if $x \in I_0$ and at $u(x) = v_1(x)$ if $x \notin I_0 \cup I_1; v_1(x) > u(x)$. In the case $x \in I_1$, the upper bound *increases* monotonically with increasing $\lambda(x)$ and thus achieves its maximum value when $u(x) = u_1(x)$. In the case $x \notin I_0 \cup I_1; v_1(x) < u(x)$ the upper bound increases monotonically with increasing $\lambda(x)$ up to a value $u(x) = v_1(x)$, but then decreases monotonically with further increases in $\lambda(x)$. Thus it achieves its maximum value at $u(x) = v_1(x)$.

In summary, the left endpoint of the subsethood interval $[s_l, s_r] \subseteq [0,1]$ is taken at the functions *u*, *v* defined by:

$$u(x) = \begin{cases} u_1(x) & x \in I_0 \\ u_0(x) & x \in I_1 \\ u_1(x) & v_1(x) \geq u_0(x) \geq v_0(x) \\ (u_0(x) \text{ or } u_1(x)) & u_0(x) < v_0(x) < u_1(x) \end{cases}. \quad (12)$$
$$v(x) = v_0(x)$$

The right endpoint s_r is taken at the functions defined by

$$u(x) = \begin{cases} u_0(x) & x \in I_0 \\ u_1(x) & x \in I_1 \\ v_1(x) & x \notin I_0, I_1 \end{cases}. \quad (13)$$
$$v(x) = v_1(x)$$

Hence

$$s_\ell = \frac{\left(\int_{x \notin I_1} v_0(x)dx + \int_{x \in I_1} u_0(x)dx\right)}{\left(\int_{x \in I_2} u_1(x)dx + \int_{x \in I_3} u_0(x)dx\right)}$$
$$s_r = \frac{\left(\int_{x \notin I_1} v_1(x)dx + \int_{x \in I_1} u_1(x)dx\right)}{\left(\int_{x \in I_0} u_0(x)dx + \int_{x \in I_1} u_1(x)dx + \int_{x \notin I_0 \cup I_1} v_1(x)dx\right)}. \quad (14)$$

Here, I_2 and I_3 are the respective indicator functions representing the conditions in the two lower expressions for $u(x)$ in (12).

B. General Type-2 Fuzzy Sets

Given Type-2 fuzzy sets $\tilde{G}$ and $\tilde{H}$, set $g_x(y) = \mu_{\tilde{G}_x}(y)$; $h_x(y) = \mu_{\tilde{H}_x}(y)$ for all $y \in [0,1]$ The approach we take to computing subsethood is similar to that used to analyse fuzzy weighted averages [16] using α-cuts of the membership functions $g_x(y)$ and $h_x(y)$. The α-cuts of $g_x(y)$ and $h_x(y)$ are the intervals

$$[u_0(x)(\alpha), u_1(x)(\alpha)] = \{y : g_x(y) \geq \alpha\},$$
$$[v_0(x)(\alpha), v_1(x)(\alpha)] = \{y : h_x(y) \geq \alpha\}. \quad (15)$$

The significance of α-cuts lies in the decomposition theorem [17], which enables a fuzzy membership function to be expressed in terms of its α-cuts:

$$g_x(y) = \sup_{\alpha \in [0,1]} \alpha I_{g_\alpha}(x). \quad (16)$$

Here, $I_{g_\alpha}(x)$ is the indicator function for the α-cuts, i.e.,

$$I_{g_\alpha}(x) = \begin{cases} 1, & x \in [u_0(x)(\alpha), u_1(x)(\alpha)] \\ 0, & otherwise \end{cases}. \quad (17)$$

The fuzzy membership functions $g_x(y)$ and $h_x(y)$ defined over the unit interval, representing the range of values taken by *y*, can be discretized into *M* α-cuts, $\alpha_1, \ldots, \alpha_M$. (Obviously, in the case of a crisp interval set, only the $\alpha = 1$ α-cut is needed.) Then by the Extension Principle, the α-cuts for a function of fuzzy variables having non-uniform set membership functions are simply given by the function applied to the α-cuts of the fuzzy variable set membership functions.

The endpoints of the intervals $[s_\ell(\alpha), s_r(\alpha)]$ can be computed for each α_m-cut of $\mu_{S(\tilde{G},\tilde{H})}(z)$ using (14). Then from (4) and the α-cut decomposition theorem [17], the Type-1 membership for $S(\tilde{G},\tilde{H})$ is given by:

$$\mu_{S(\tilde{G},\tilde{H})}(z) = \sup_{\forall \alpha \in [0,1]} \begin{cases} \alpha, & s_\ell(\alpha) \leq z \leq s_r(\alpha) \\ 0 & elsewhere \end{cases}. \quad (18)$$

In other words, the α-cuts of $\mu_{S(\tilde{G},\tilde{H})}(z)$ are determined by the intervals $[s_\ell(\alpha), s_r(\alpha)]$. Once the points $(z_{\ell m}, \alpha_m)$ and (z_{rm}, α_m) corresponding to each of the discrete α_m-cuts have been computed using (15) - (18), the intermediate values may be interpolated from these points.

IV. Subsethood of Type-3 fuzzy sets with interval Type-2 secondary membership

To compute this subsethood, we use an approach analogous to that of Wu and Mendel [18] for linguistic weighted averages.

Suppose $\mu_{\tilde{G}_x^3}$ and $\mu_{\tilde{H}_x^3}$ are membership functions of interval Type-2 fuzzy sets associated with the Type-3 fuzzy sets $\tilde{G}^{(3)}, \tilde{H}^{(3)}$ respectively (cf. Fig. 1). Denote the upper and lower functions membership functions that bound the FOUs by $g_x^U(y)$ and $g_x^L(y)$, and $h_x^U(y)$ and $h_x^L(y)$, respectively. By the proposition in section 2, $S\left(\tilde{G}^{(3)}, \tilde{H}^{(3)}\right)$ has an interval Type-2 membership function $\mu_{S\left(\tilde{G}^{(3)}, \tilde{H}^{(3)}\right)}(z,w)$ on $[0,1]$. Let $\mu^U(z)$ denote the upper bound of the subsethood interval $\left\{w : \mu_{\left(\tilde{G}^{(3)}, \tilde{H}^{(3)}\right)}(z,w) = 1\right\}$ and $\mu^L(z)$ denote the lower bound.

From (5), $\mu_{S\left(\tilde{G}^{(3)}, \tilde{H}^{(3)}\right)}(z,w) = 1$ if and only if there exist functions $u, u', v, v' : X \to [0,1]$ satisfying the following conditions:

$f(u,v) = z, \quad \inf_{x \in X}\left(u'(x), v'(x)\right) = w$;

$g_x^L\left(u(x)\right) \le u'(x) \le g_x^U\left(u(x)\right)$ for all $x \in X$; and

$h_x^L\left(v(x)\right) \le v'(x) \le h_x^U\left(v(x)\right)$ for all $x \in X$.

Subsethood membership is zero otherwise.

Define Type-2 fuzzy sets $\tilde{G}_U, \tilde{G}_L, \tilde{H}_U, \tilde{H}_L$ to have the membership functions

$$\begin{aligned} \mu_{\tilde{G}_U}(x,y) &= g_x^U(y), \\ \mu_{\tilde{G}_L}(x,y) &= g_x^L(y) \end{aligned} \tag{19}$$

and similarly for $\mu_{\tilde{H}_U}(x,y)$ and $\mu_{\tilde{H}_L}(x,y)$. From the definitions of subsethood (5) and of the membership functions (19),

$$\begin{aligned} \mu_{S\left(\tilde{G}_U, \tilde{H}_U\right)}(z) &= \sup_{u,v: f(u,v)=z} \inf_{x \in X}\left\{\mu_{\tilde{G}_U}\left(x, u(x)\right), \mu_{\tilde{H}_U}\left(x, v(x)\right)\right\} \\ &= \sup_{f(u,v)=z} \inf_{x \in X}\left\{g_x^U\left(u(x)\right), h_x^U\left(v(x)\right)\right\} \end{aligned} \tag{20}$$

and similarly for $S\left(\tilde{G}_L, \tilde{H}_L\right)$. Moreover, we know from the previous section how to compute $S\left(\tilde{G}_U, \tilde{H}_U\right)$ and $S\left(\tilde{G}_L, \tilde{H}_L\right)$

By definition of $g_x^U(y)$, etc. as the bounds of the FOUs,

$\inf_{x \in X}\left\{\mu_{\tilde{G}^{(3)}}\left(x, u(x), g_x^U\left(u(x)\right)\right), \mu_{\tilde{H}^{(3)}}\left(x, v(x), h_x^U\left(v(x)\right)\right)\right\} = 1$.

So $\mu_{S\left(\tilde{G}_U, \tilde{H}_U\right)}(z) \le \mu^U(z)$ and likewise $\mu_{S\left(\tilde{G}_L, \tilde{H}_L\right)}(z) \ge \mu^L(z)$. However, for $u', v' : X \to [0,1]$, $\mu_{\tilde{G}^{(3)}}\left(x, u(x), u'(x)\right) = 1$ if and only if $g_x^L\left(u(x)\right) \le u'(x) \le g_x^U\left(u(x)\right)$ and likewise $\mu_{\tilde{H}^{(3)}}\left(x, v(x), v'(x)\right) = 1$ if and only if $h_x^L\left(v(x)\right) \le v'(x) \le h_x^U\left(v(x)\right)$. Thus

$$\mu_{S\left(\tilde{G}_U, \tilde{H}_U\right)}(z) \equiv \sup_{f(u,v)=z} \inf_{x \in X}\left\{g_x^U\left(u(x)\right), h_x^U\left(v(x)\right)\right\}$$

$$\ge \sup_{\substack{f(u,v)=z \\ g^L(u(x)) \le u'(x) \le g^U(u(x)) \\ h^L(v(x)) \le v'(x) \le h^U(v(x))}} \inf_{x \in X}\left\{u'(x), v'(x)\right\} \equiv \mu^U(z).$$

So $\mu_{S\left(\tilde{G}_U, \tilde{H}_U\right)}(z) = \mu^U(z)$. Likewise $\mu_{S\left(\tilde{G}_L, \tilde{H}_L\right)}(z) = \mu^L(z)$. The region lying between these bounding functions is the FOU of $S\left(\tilde{G}^{(3)}, \tilde{H}^{(3)}\right)$, i.e.

$$\mu_{S\left(\tilde{G}^{(3)}, \tilde{H}^{(3)}\right)}(z,w) = \begin{cases} 1 & \mu^L(z) \le w \le \mu^U(z) \\ 0 & \textit{elsewhere} \end{cases} \tag{21}$$

V. Examples of Subsethood

A. General Type-2 fuzzy sets

Consider a set X having two elements, on which is defined two Type-2 fuzzy sets with truncated Gaussian exponential Type-1 secondary membership functions

$$\mu(y) = \begin{cases} \exp\left(-\dfrac{(y-m_i)^2}{2\sigma_i^2}\right), & -3 \le \dfrac{y-m_i}{\sigma_i} \le 3,\ 0 \le y \le 1 \\ 0, & \textit{elsewhere} \end{cases}. \tag{22}$$

Let the individual fuzzy membership functions in (22) have the following parameters:

$\mu_{G_1}(y)$: $m_{G_1} = 0.25$; $\sigma_{G_1} = 0.03$

$\mu_{G_2}(y)$: $m_{G_2} = 0.6$; $\sigma_{G_2} = 0.02$

$\mu_{H_1}(y)$: $m_{H_1} = 0.15$; $\sigma_{H_1} = 0.01$

$\mu_{H_2}(y)$: $m_{H_2} = 0.3$; $\sigma_{H_1} = 0.02$

Then using (18), compute the corresponding Type-1 membership function for subsethood. Fig. 2 shows the input secondary membership functions with these parameters (represented by the solid curves) along with the results of this calculation (dashed curve).

B. Subsethood of Type-3 fuzzy sets with interval Type-2 secondary membership functions

Consider two interval Type-3 fuzzy sets, where the Type-2 secondary membership functions have upper and lower membership functions of the form of (22), with the means m_i the same for each pair, but with a larger σ_i for the upper membership function. Fig. 3 shows these functions, with the area between these functions being the footprint of uncertainty.

Using (21), we can calculate the corresponding Type-2 fuzzy set membership for subsethood. Fig. 4 shows the upper and lower membership functions, where the area between the

solid and dashed curves represents the footprint of uncertainty for each function.

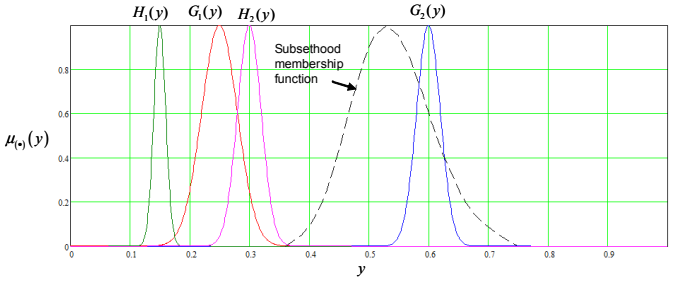

Fig.2. Type-1 secondary membership functions of general Type-2 fuzzy sets and corresponding Type-1 membership function for subsethood of $\tilde{G}$ in $\tilde{H}$.

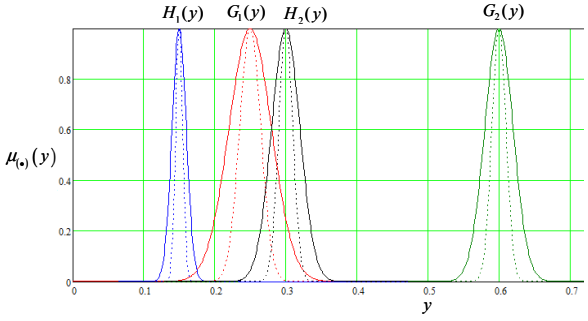

Fig. 3 Type-2 secondary fuzzy membership functions of Type-3 fuzzy sets. The regions between the solid and dashed curves are the footprints of uncertainty.

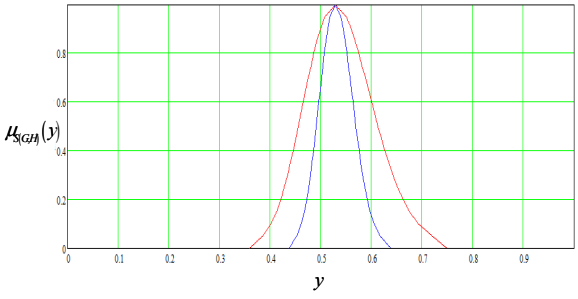

Fig. 4. Type-2 fuzzy membership function for the subsethood of $\tilde{G}$ and $\tilde{H}$ for the input Type-2 secondary fuzzy membership functions shown in Figure 3. The region between these two curves is the footprint of uncertainty.

VI. Conclusion

We defined fuzzy subsethood for the case of general Type-n fuzzy sets defined on any set X with a measure. The Type definition follows from the application of Zadeh's Extension Principle to Kosko's definition of Type-1 subsethood. The general Type-n definitions follow from recursive application of the Extension Principle. We showed how to compute the membership as a function of the endpoints of the α-cuts in the case of Type-2 fuzzy sets, and gave illustrations for Type-2 and interval Type-3 fuzzy sets.

As for the case of subsethood of Type-1 fuzzy sets, there will be many applications for the generalised subsethood. For example, color is often imprecisely gauged due to lighting conditions and so on, and thus might be represented by assigning Type-2 fuzzy values on three dimensions (e.g. RGB). Subsethood might then be used to describe the relationship between colors, generalizing the approach in [6].

As with subsethood of Type-1 fuzzy sets, there are also plausible alternative definitions of the generalised subsethood. A future analysis is needed of features and applicability of various contenders.

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